MONOCHROMATIC GENERATING SETS IN GROUPS AND OTHER ALGEBRAIC STRUCTURES

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Abstract. The *generating chromatic number* of a group G, $\chi_{\mathrm{gen}}(G)$, is the maximum number of colors k such that there is a monochromatic generating set for each coloring of the elements of G in k colors. If no such maximal k exists, we set $\chi_{\mathrm{gen}}(G)=\infty$. Equivalently, $\chi_{\mathrm{gen}}(G)$ is the maximal number k such that there is no cover of G by proper subgroups (∞ if there is no such maximal k).

We provide characterizations, for arbitrary gruops, in the cases $\chi_{\mathrm{gen}}(G)=\infty$ and $\chi_{\mathrm{gen}}(G)=2$. For nilpotent groups (in particular, for abelian ones), all possible chromatic numbers are characterized. Examples show that the characterization for nilpotent groups do not generalize to arbitrary solvable groups. We conclude with applications to vector spaces and fields.

Remark. After completing this paper, we have learned that earlier works, using different terminology, established most—if not all—of our results. We chose not to remove our proofs, but to provide references to all earlier proofs. Our Theorem 3.1 may be new, but it is not of great distance from known results, either.

1. General properties and the case of infinite chromatic number

Results in this section with no reference are folklore. All group theoretic background used in this paper can be found, for example, in [3]. We study the following Ramsey theoretic notion.

Definition 1.1. The *generating chromatic number* of a group G, $\chi_{\mathrm{gen}}(G)$, is the maximum number of colors k such that there is a monochromatic generating set for each coloring of the elements of G in k colors. If no such maximal k exists, we set $\chi_{\mathrm{gen}}(G)=\infty$.

The following lemma gives a convenient reformulation of our definition.

Lemma 1.2. *Let G be a group. The following are equivalent:*

1. $\chi_{\mathrm{gen}}(G)\geq k$.
2. *For each cover $G=H_1\cup\cdots\cup H_k$ of G by subgroups, there is i with $H_i=G$.*

Proof. The first implication is obtained by giving elements of $H_i\setminus(H_1\cup\cdots\cup H_{i-1})$ the color i, for each $i=1,\ldots,k$.

For the second implication, replace each maximal monochromatic set by the subgroup it generates. □

Lemma 1.3. *Let G be a group.*

1. $\chi_{\mathrm{gen}}(G)\geq 2$.
2. *If G is cyclic, then $\chi_{\mathrm{gen}}(G)=\infty$.*
3. *If G is finite noncyclic, then $\chi_{\mathrm{gen}}(G)\leq |G|-2$.*

Proof. (1) This is well known: If $G = H_1 \cup H_2$ with H_1, H_2 proper subgroups, then for $x \in G \setminus H_1 \subseteq H_2$ and $y \in G \setminus H_2 \subseteq H_1$, we have that $xy \notin H_1 \cup H_2$, a contradiction.

(2) A generator forms a monochromatic generating subset for each coloring.

(3) Give each element of $G \setminus \{e\}$ a unique color. $\square$

Cyclic groups are not the only abelian groups with infinite generating chromatic number.

Example 1.4. $\chi_{\mathrm{gen}}(\mathbb{Q}, +) = \infty$.

Proof. This follows from the forthcoming Theorem 1.12; however, we give a direct proof.

Assume that $\mathbb{Q} = H_1 \cup \cdots \cup H_k$ with each H_i a subgroup of $(\mathbb{Q}, +)$. Then there is i such that H_i contains infinitely many rationals of the form $\frac{1}{n!}$. Then $H_i = \mathbb{Q}$. Indeed, for each rational $\frac{l}{m}$, pick $n \geq m$ with $\frac{1}{n!} \in H_i$. Then $\frac{n!}{m}$ is integer, and therefore

$$\frac{l}{m} = \frac{n!l}{m} \cdot \frac{1}{n!} \in H_i.$$

$\square$

Lemma 1.5 (Cohn [2])**.** *If H is a quotient (equivalently, a homomorphic image) of a group G, then $\chi_{\mathrm{gen}}(G) \leq \chi_{\mathrm{gen}}(H)$.*

Proof. Let $\varphi \colon G \to H$ be a surjective homomorphism. Give each $g \in G$ the color of $\varphi(g)$. Then a monochromatic generating subset of G would map to a monochromatic generating subset of H. $\square$

Definition 1.6. A cover $G = H_1 \cup \cdots \cup H_k$ of a group G by subgroups is *irredundant* if, for each $J \subsetneq \{1, \dots, k\}$, $G \neq \bigcup_{i \in J} H_i$.

Lemma 1.7. *Let G be a group and k be a natural number. Then:*

1. *If $\chi_{\mathrm{gen}}(G) = k$, then there is an irredundant cover $G = H_1 \cup \cdots \cup H_{k+1}$ of G by subgroups.*
2. *If there is an irredundant cover $G = H_1 \cup \cdots \cup H_{k+1}$ of G by subgroups, then $\chi_{\mathrm{gen}}(G) \leq k$.* $\square$

The following theorem is proved in [4] for finite groups, but the proof actually treats the case of torsion groups.

Proposition 1.8 (Haber–Rosenfeld [4])**.** *Let G be a finite (or just torsion) group, and p be the minimal order of an element of G. Then $\chi_{\mathrm{gen}}(G) \geq p$.*

Proof. Assume that $k := \chi_{\mathrm{gen}}(G) < \infty$. By Lemma 1.3, $k \geq 2$.

Let $G = H_1 \cup \cdots \cup H_{k+1}$ be an irredundant cover by subgroups. Pick

$$g_1 \in H_1 \setminus (H_2 \cup \cdots \cup H_{k+1}), g_2 \in H_2 \setminus (H_1 \cup H_3 \cup \cdots \cup H_{k+1}).$$

For each $0 \leq n < p$, as $g_1^n \in H_1$ and $g_2 \notin H_1$, we have that$g_1^n g_2 \notin H_1$.

Assume that $p > k$. Then, by the pigeonhole principle, there are $0 \leq m < n < p$ and $2 \leq i \leq k$ such that $g_1^m g_2, g_1^n g_2 \in H_i$. Thus, $g_1^{n-m} \in H_i$. As the minimal order of an element of G is p and $1 \leq n - m < p$, g_1 is a power of g_1^{n-m}, and thus $g_1 \in H_i$, a contradiction. $\square$

Proposition 1.9 (Cohn [2])**.** *Let A, B be finite (or just torsion) groups such that the order of each element of A is coprime to the order of each element of B. Then $\chi_{\mathrm{gen}}(A \times B) = \min\{\chi_{\mathrm{gen}}(A), \chi_{\mathrm{gen}}(B)\}$.*

Proof. ($\leq$) By Lemma 1.5.

($\geq$) Let $k \leq \min\{\chi_{\mathrm{gen}}(A), \chi_{\mathrm{gen}}(B)\}$, and let $c\colon A \times B \to \{1, \dots, k\}$ be a coloring of the elements of $A \times B$ in the colors $\{1, \dots, k\}$. For each $a \in A$, define $c_a\colon B \to \{1, \dots, k\}$ by

$$c_a(b) = c(a, b).$$

As $k \leq \chi_{\mathrm{gen}}(B)$, there is a monochromatic generating set M_a of B, of some color i_a. Now, define a coloring $f\colon A \to \{1, \dots, k\}$ by

$$f(a) = i_a.$$

Let $M \subseteq A$ be a monochromatic generating set, say of color i. Let

$$I = \{(a, b) \in A \times B \,:\, c(a, b) = i\}.$$

We claim that I generates $A \times B$. Indeed, fix $a \in M$ and $b \in M_a$. Then $(a, b) \in I$. Let m be the order of b. As m is coprime to the order of a, (a, e) is a power of$(a^m, e) = (a, b)^m$, and thus belongs to $\langle I \rangle$. Similarly, $(e, b) \in \langle I \rangle$. It follows that $M \times \{e\}, \{e\} \times M_a \subseteq \langle I \rangle$, and therefore $A \times \{e\}, \{e\} \times B \subseteq \langle I \rangle$. □

The following result, proved in [6](4.4), will make it possible for us to reduce the case of arbitrary groups into the case of finite ones.

Theorem 1.10 (Neumann)**.** *For each irredundant cover $G = H_1 \cup \cdots \cup H_k$ of a group G by subgroups, each of the subgroups $H_1, \dots, H_k$ has finite index in G.*

Corollary 1.11. *Let G be a group with $\chi_{\mathrm{gen}}(G) = k < \infty$. Let $G = H_1 \cup \cdots \cup H_{k+1}$ be an irredundant cover of G by subgroups. There is a normal subgroup N of G such that:*

1. *N has finite index in G;*
2. *$N \subseteq H_1 \cap \cdots \cap H_k$;*
3. *$G/N = H_1/N \cup \cdots \cup H_k/N$ is an irredundant cover of the finite group G/N; and*
4. *$\chi_{\mathrm{gen}}(G/N) = k$.*

Proof. (1,2) By Neumann's Theorem 1.10, the subgroup $H := H_1 \cap \cdots \cap H_{k+1}$, being an intersection of finite index subgroups of G, has finite index in G. By Poincaré's Theorem, there is a subgroup N of H such that N is a finite index, normal subgroup of G.

(3) For $J \subsetneq \{1, \dots, k+1\}$, Let $g \in G \setminus \bigcup_{i \in J} H_i$. If $gN \in \bigcup_{i \in J} H_i/N$, say $gN \in H_i/N$, then $g \in h_i N \subseteq H_i$, a contradiction.

(4) By Lemma 1.5, Lemma 1.7 and (3), $k = \chi_{\mathrm{gen}}(G) \leq \chi_{\mathrm{gen}}(G/N) \leq k$. □

We already obtain a characterization of the case where $\chi_{\mathrm{gen}}(G) = \infty$.

Theorem 1.12. *Let G be a group. The following assertions are equivalent:*

1. *$\chi_{\mathrm{gen}}(G) = \infty$.*
2. *Every finite quotient of G is cyclic.*

Proof. ($1 \Rightarrow 2$) Let G/N be a finite quotient of G. By Lemma 1.5, $\chi_{\mathrm{gen}}(G/N) = \infty$. By Lemma 1.3, G is cyclic.

($2 \Rightarrow 1$) Assume that $\chi_{\mathrm{gen}}(G) < \infty$. Then, by Corollary 1.11, some finite quotient of G has the same finite generating chromatic number, and is thus not cyclic. □

It follows, for example, that infinite simple groups (e.g., A_∞ or $\mathrm{PSL}_n(\mathbb{F})$, $n \geq 3$) have infinite generating chromatic number.

2. Nilpotent groups of finite chromatic number

Lemma 2.1 (Cohn [2]). $\chi_{\mathrm{gen}}(\mathbb{Z}_p \times \mathbb{Z}_p) = p$.

Proof. ($\geq$) Proposition 1.8.

($\leq$) $\mathbb{Z}_p \times \mathbb{Z}_p$ is the union of $p+1$ projective lines, the cyclic subgroups generated by the $p+1$ elements $(0,1)$ and $(1,0),(1,1),\dots,(1,p-1)$. □

Say that a group G is *projectively nilpotent* if every finite quotient of G is nilpotent. In particular, every nilpotent, quasi-nilpotent, or pro-nilpotent group is projectively nilpotent. In the finite case, the following theorem was proved by Cohn [2]. The infinite case follows from Corollary 1.11, which in turns follows from Neumann's Theorem 1.10.

Theorem 2.2 (Cohn, Neumann). *Let G be a projectively nilpotent group with $\chi_{\mathrm{gen}}(G) < \infty$. Then $\chi_{\mathrm{gen}}(G)$ is the minimum prime number p such that $\mathbb{Z}_p \times \mathbb{Z}_p$ is a quotient of G (and there is such p).*

Proof. Let G be a projectively nilpotent group with $\chi_{\mathrm{gen}}(G) < \infty$. By Corollary 1.11, we may assume that G is a finite nilpotent group. Write

$$G = P_1 \times \cdots \times P_n,$$

where each P_i is the p_i-Sylow subgroup of G. By Lemma 1.9,

$$\chi_{\mathrm{gen}}(G) = \min\left\{\chi_{\mathrm{gen}}(P_1),\dots,\chi_{\mathrm{gen}}(P_n)\right\}.$$

Let $P = P_i$ be with $\chi_{\mathrm{gen}}(G) = \chi_{\mathrm{gen}}(P_i)$, and $p = p_i$.

By Lemma 1.8, $\chi_{\mathrm{gen}}(P) \geq p$. As P is a noncyclic, its rank r is greater than 1. As P is a p-group, $\mathbb{Z}_p^r$ is a quotient of P, and in particular of G.

Thus, $\chi_{\mathrm{gen}}(G) = \chi_{\mathrm{gen}}(P) = p$ and $\mathbb{Z}_p \times \mathbb{Z}_p$ is a quotient of G. By Lemmata 1.5 and 2.1, there is no smaller prime q such that $\mathbb{Z}_q \times \mathbb{Z}_q$ is a quotient of G. □

3. General groups

The forthcoming Theorem may be new. Responding to a question of us, Martino Garonzi came up with two alternative proofs of Theorem . One of these proofs uses a powerful result of Tomkinson [8, Lemma 3.2].[1] We are greatful to Martino Garonzi for this information.

We will show, in Section 4, that Theorem 2.2 does not generalize to arbitrary (or even finite solvable) groups. In the present section, we provide a partial generalization of this theorem to arbitrary torsion groups. Recall, from Lemma 1.8, that the premise in the following theorem implies that $\chi_{\mathrm{gen}}(G) \geq p$.

Theorem 3.1. *Let G be a nontrivial torsion group, and let p be the minimal order of a nonidentity element of G. The following assertions are equivalent:*

(1) $\chi_{\mathrm{gen}}(G) = p$.
(2) $\mathbb{Z}_p \times \mathbb{Z}_p$ *is a quotient of* G.

[1] Garonzi also points a misprint there: It is necessary to assume there that the union of the U_i's is different from G.

Proof. (2) $\Rightarrow$ (1): Proposition 1.8, Lemma 1.5 and Lemma 2.1.

(1) $\Rightarrow$ (2): Assume that G/N is a quotient of G. The order of each element $gN \in G/N$ is $\geq p$. Indeed, the order of gN divides the order of g, and is therefore the order of a power of g, which in turn is $\geq p$. Thus, let

$$G = H_1 \cup \cdots \cup H_{p+1}$$

be an irredundant cover by proper subgroups. Then, by Corollary 1.11, we know G has a finite quotient G/N with $\chi_{\text{gen}}(G/N) = \chi_{\text{gen}}(G)$. Since each element of G/N is of order $\geq p$ we may assume G is finite. We now have by Cauchy's theorem that $|G|$ and i are relatively prime for each $i < p$.

Let H be a subgroup of G. Then $[G : H]$ divides $|G|$, and by Cauchy's theorem, there is $g \in G$ whose order divides $[G : H]$. In particular, $[G : H] \geq p$ and $|H| \leq |G|/p$. Assume that, for each $i = 1, \ldots, p+1$, $[G : H_i] \geq p+1$. Then, as $e \in H_1 \cap \cdots \cap H_{p+1}$,

$$|G| = |H_1 \cup \cdots \cup H_{p+1}| < |H_1| + \cdots + |H_{p+1}| \leq (p+1) \cdot \frac{|G|}{p+1} = |G|,$$

a contradiction. Thus, we may assume that $[G : H_1] = p$. As p is the minimal prime divisor of $|G|$, H_1 is a *normal* subgroup of G.

Fix $i \in \{2, \ldots, p+1\}$. Then $p = [G : H_1] = [G : H_i H_1] \cdot [H_i H_1 : H_1]$. As p is prime and $H_i \neq H_1$ we have by the Second Isomorphism Theorem that $p = [H_i H_1 : H_1] = [H_i : H_i \cap H_1]$. It follows that

$$|H_i \setminus H_1| = |H_i \setminus (H_i \cap H_1)| = \frac{p-1}{p} \cdot |H_i|.$$

Thus,

$$\begin{aligned}
\frac{p-1}{p} \cdot |G| &= |G \setminus H_1| = |(H_2 \setminus H_1) \cup (H_3 \setminus H_1) \cup \cdots \cup (H_{p+1} \setminus H_1)| \leq \\
&\leq |H_2 \setminus H_1| + |H_3 \setminus H_1| + \cdots + |H_{p+1} \setminus H_1| = \\
&= \frac{p-1}{p}(|H_2| + |H_3| + \cdots + |H_{p+1}|).
\end{aligned}$$

As $|H_i| \leq |G|/p$ for each i,

$$|G| \leq |H_2| + \cdots + |H_{p+1}| \leq \frac{|G|}{p} + \cdots + \frac{|G|}{p} = |G|.$$

Thus $|H_2| = |H_3| = \cdots = |H_{p+1}| = |G|/p$, and therefore $H_2, \ldots, H_{p+1}$ are also normal subgroups of G.

Let $N = H_1 \cap H_2 \cap \cdots \cap H_{p+1}$. By moving to $G/N = H_1/N \cup \cdots \cup H_{p+1}/N$ if needed, we may assume that $H_1 \cap \cdots \cap H_{p+1} = \{e\}$.

We claim that all elements of G are of order p.

Let $g \in G$ be with $g^p \neq e$. We may assume that $g \in H_1$. Then $g^p \in H_1$. As $H_1 \cap \cdots \cap H_{p+1} = \{e\}$, we may assume that $g^p \notin H_2$. Let $h \in H_2 \setminus (H_1 \cup H_3 \cup \cdots \cup H_{p+1})$. At least two of the $p+2$ elements

$$g, h, hg, hg^2, \ldots, hg^p$$

are in the same H_i.

If $i = 1$, then $g, hg^j \in H_1$ for some j and therefore $h \in H_1$, a contradiction.

If $i = 2$, then similarly $g^j \in H_2$ for some $j \in \{0, \ldots, p\}$. As $g^p \notin H_2$, $j < p$. By Cauchy's theorem, p is the minimal prime factor of $|G|$. Thus, j is coprime to $|G|$, and in particular to the order of g. Thus, $g \in \langle g^j \rangle \subseteq H_2$, and therefore $g^p \in H_2$, a contradiction.

Thus, $i > 2$. We may assume that $i = 3$. If $g, hg^j \in H_3$ for some j, then $h \in H_3$, a contradiction. Thus, $hg^j, hg^k \in H_3$ for some $0 \leq j < k \leq p$. Then $g^{k-j} \in H_3$. As $k - j < p$, $g \in H_3$ and thus also $h \in H_3$, a contradiction.

Thus, all elements of G are of order p, and in particular G is a p-group. As $\chi_{\text{gen}}(G) = p < \infty$, G is not cyclic, and therefore $\mathbb{Z}_p \times \mathbb{Z}_p$ is a quotient of G. □

In the case of 2 colors, we obtain a completely general characterization.

Theorem 3.2 (Scorza [7])**.** *Let G be a group. The following assertions are equivalent:*

(1) $\chi_{\text{gen}}(G) = 2$.
(2) $\mathbb{Z}_2 \times \mathbb{Z}_2$ *is a quotient of* G.

Proof. ($2 \Rightarrow 1$) Lemmata 1.5, 2.1, and 1.3.

($1 \Rightarrow 2$) By Corolllary1.11, we may assume that G is a nontrivial finite group. By Proposition 1.8, the minimal order of a nonidentity element of G is 2. Apply Theorem 3.1. □

We will show, in Example 4.7 below, that a 3-colors version of Theorem 3.2 is not true in general.

4. Examples

All results in this section can, alternatively, be derived from Tomkinson's treatment of the general finite solvable case [8].

One may wonder whether Theorem 2.2 holds for arbitrary groups G, or at least for arbitrary solvable groups. We will show that the answer is negative, in a strong sense.

Recall that the *Frattini subgroup* of a group G, $\phi(G)$, is defined to be the intersection of all maximal subgroups of G, or G if G has no maximal subgroups. Recall that the elements of $\phi(G)$ are all the non-generators of G, that is, all elements $g \in G$ such that $G = \langle g, H \rangle$ implies $H = G$.

Lemma 4.1. *Let $H \trianglelefteq G$ be a subgroup of the Frattini subgroup $\phi(G)$. Then $\chi_{\text{gen}}(G) = \chi_{\text{gen}}(G/H)$.*

Proof. ($\leq$) By Lemma 1.5.

($\geq$) Let $k = \chi_{\text{gen}}(G)$, and let $G = H_1 \cup H_2 \cup \cdots \cup H_{k+1}$ be an irredundant cover of G by subgroups. As each H_i is of finite index, we can replace each H_i by a group of minimal index containing H_i. Thus, for each i, we may asume that H_i is maximal, and therefore $H \leq H_i$. As

$$G/H = H_1/H \cup H_2/H \cup \cdots \cup H_{k+1}/H,$$

we have that $\chi_{\text{gen}}(G/H) \leq k$. □

Lemma 4.2. $\chi_{\text{gen}}(\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_n) = p$ *if φ is nontrivial. In general, $\chi_{\text{gen}}(\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_n)$ is either p or ∞.*

Proof. Write $\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_n = \langle x \rangle \rtimes \langle y \rangle$. $\langle x \rangle$ acts on the subgroups of $\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_n$ by conjugation. We claim that $|[\langle y \rangle]| = p$. Since $\langle y \rangle$ is maximal non-normal, $N_{\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_n}(\langle y \rangle) = \langle y \rangle$ and therefore $N_{\langle x \rangle}(\langle y \rangle) = \langle y \rangle \cap \langle x \rangle = \{1\}$. Thus

$$|[\langle y \rangle]| = \left[\langle x \rangle : N_{\langle x \rangle}(\langle y \rangle)\right] = [\langle x \rangle : \{1\}] = p.$$

Since $(\langle y \rangle \cap \langle y^x \rangle)^x$ is a subgroup of the cyclic group $\langle y^x \rangle$, with the same order as $\langle y \rangle \cap \langle y^x \rangle$, we have that $\langle y \rangle \cap \langle y^x \rangle = (\langle y \rangle \cap \langle y^x \rangle)^x$. Thus, $x, y \in N_{\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_n}(\langle y \rangle \cap \langle y^x \rangle)$, and therefore $\langle y \rangle \cap \langle y^x \rangle \trianglelefteq \mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_n$. Let $m = |\langle y \rangle \cap \langle y^x \rangle|$. Then $\langle y \rangle \cap \langle y^x \rangle = \left\langle y^{\frac{n}{m}} \right\rangle$. Since $\left[x, y^{\frac{n}{m}}\right] \in \left\langle y^{\frac{n}{m}} \right\rangle \cap \langle x \rangle = 1$, $y^{\frac{n}{m}}$ commutes with both x and y, and therefore $y^{\frac{n}{m}} \in Z(\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_n)$. Let $k = \chi_{\text{gen}}(\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_n)$, and let

$$\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_n = H_1 \cup H_2 \cup \cdots \cup H_{k+1}$$

be an irredundant cover of $\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_n$ by subgroups. Since each element $A \in [\langle y \rangle]$ is cyclic and maximal,

$$[\langle y \rangle] \subseteq \{H_1, H_2, \ldots, H_{k+1}\}.$$

Assume that $[\langle y \rangle] = \{H_1, H_2, \ldots, H_{k+1}\}$. Then $x \in H_1 \cup H_2 \cup \cdots \cup H_{k+1} = \bigcup_{g \in \langle x \rangle} \langle g^{-1} y g \rangle$, and therefore $x = g^{-1} y^m g$ for some $g \in \langle x \rangle, m \in \mathbb{Z}$, and thus $gxg^{-1} \in \langle y \rangle \cap \langle x \rangle = \{1\}$, a contradiction. Thus, $p \leq k$.

To prove that $p \geq k$, let $H_i = \left\langle y^{\left(x^i\right)} \right\rangle$ for $i = 1, \ldots, p$, and let $H_{p+1} = \langle x \rangle \left\langle y^{\frac{n}{m}} \right\rangle$.

We claim that all intersections $H_i \cap H_j$, $i \neq j$, are contained in $\left\langle y^{\frac{n}{m}} \right\rangle$. First, let $i, j \leq p$. As in the case $i = 1, j = 2$, we have that $H_i \cap H_j \trianglelefteq \mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_n$, and therefore $H_i \cap H_j \leq H_i^g$ for each $g \in \langle x \rangle$. In particular, $H_i \cap H_j \leq H_1 \cap H_2 = \langle y \rangle \cap \langle y^x \rangle = y^{\frac{n}{m}}$. Next, let $i < j = p+1$. Assume that $g \in H_i \cap H_{p+1}$. Then

$${}^{x^i}g \in {}^{x^i} H_i \cap H_{p+1} = H_1 \cap H_{p+1}.$$

Write ${}^{x^i}g = x^a y^{\frac{nb}{m}}$. Then $x^a y^{\frac{nb}{b}} \in \langle y \rangle$, and therefore $x^a \in \langle y \rangle \cap \langle x \rangle = \{1\}$. Thus, ${}^{x^i}g = y^{\frac{nb}{m}}$. Since $y^{\frac{nb}{m}}$ is in the center of $\mathbb{Z}_{p^n} \rtimes_\varphi \mathbb{Z}_m$, we have that $g \in \left\langle y^{\frac{n}{m}} \right\rangle$. Thus,

$$\begin{aligned}
&\left|H_1 \setminus \left\langle y^{\frac{n}{m}} \right\rangle \cup H_2 \setminus \left\langle y^{\frac{n}{m}} \right\rangle \cup \cdots \cup H_{p+1}\right| = \\
&= \left|H_1 \setminus \left\langle y^{\frac{n}{m}} \right\rangle\right| + \left|H_2 \setminus \left\langle y^{\frac{n}{m}} \right\rangle\right| + \ldots + |H_{p+1}| \geq \\
&\geq \left(n - \frac{n}{m}\right) p + \frac{pn}{m} = np.
\end{aligned}$$

As

$$H_1 \setminus \left\langle y^{\frac{n}{m}} \right\rangle \cup H_2 \setminus \left\langle y^{\frac{n}{m}} \right\rangle \cup \cdots \cup H_{p+1} \subseteq H_1 \cup H_2 \cup \cdots \cup H_{p+1},$$

we have that $|H_1 \cup H_2 \cup \cdots \cup H_{p+1}| \geq np$, and therefore $H_1 \cup H_2 \cup \cdots \cup H_{p+1} = \mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_n$.

Finally, if φ is trivial, then $\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_n$ is abelian and Theorem 2.2 applies. □

Lemma 4.3. *Let n be a natural number and p be a prime number. If φ is non-trivial, then $\chi_{\text{gen}}(\mathbb{Z}_{p^n} \rtimes_\varphi \mathbb{Z}_m) = p$.*

Proof. Write $\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_n = \langle x \rangle \rtimes \langle y \rangle$. We first show $\langle x^p \rangle \leq \phi(\mathbb{Z}_{p^n} \rtimes_\varphi \mathbb{Z}_m)$. Let H be maximal in $\mathbb{Z}_{p^n} \rtimes_\varphi \mathbb{Z}_m$ then $H \langle x^p \rangle$ is a subgroup containing H. Since $\langle x^p \rangle \trianglelefteq \mathbb{Z}_{p^n} \rtimes_\varphi \mathbb{Z}_m$, $H \leq H \langle x^p \rangle \leq \mathbb{Z}_{p^n} \rtimes_\varphi \mathbb{Z}_m$. Assume that $\langle x^p \rangle \not\leq H$. Then $H \langle x^p \rangle = \mathbb{Z}_{p^n} \rtimes_\varphi \mathbb{Z}_m$, and therefore $x = hx^{ap}$ for some $h \in H, a \in \mathbb{Z}$. Thus, $x^{1-ap} \in H$. Since $1 - ap$ and p^n are relatively prime, we have that $x \in H$, a contradiction. Thus, $\langle x^p \rangle \leq H$.

By Lemma 4.1,

$$\chi_{\text{gen}}\left(\mathbb{Z}_{p^n} \rtimes_\varphi \mathbb{Z}_m\right) = \chi_{\text{gen}}\left(\mathbb{Z}_{p^n} \rtimes_\varphi \mathbb{Z}_m / \langle x^p \rangle\right) = \chi_{\text{gen}}\left(\mathbb{Z}_p \rtimes_{\varphi'} \mathbb{Z}_m\right) \in \{p, \infty\}.$$

Since $\mathbb{Z}_{p^n} \rtimes_\varphi \mathbb{Z}_m$ is non-abelian, and therefore non-cyclic, $\chi_{\text{gen}}(\mathbb{Z}_{p^n} \rtimes_\varphi \mathbb{Z}_m) < \infty$, and therefore $\chi_{\text{gen}}\left(\mathbb{Z}_{p^n} \rtimes_\varphi \mathbb{Z}_m\right) = p$. $\square$

Corollary 4.4. $\chi_{\text{gen}}\left(\mathbb{Z}_{p^n} \rtimes_\varphi \mathbb{Z}_m\right) = \infty$ *if m, p are relatively prime and φ is trivial. Otherwise,* $\chi_{\text{gen}}\left(\mathbb{Z}_{p^n} \rtimes_\varphi \mathbb{Z}_m\right) = p$.

Proof. Lemmata 2.2 and 4.3. $\square$

Theorem 4.5. *Let $G = \mathbb{Z}_m \rtimes_\varphi \mathbb{Z}_n$, and write $\mathbb{Z}_m = \mathbb{Z}_{p_1^{d_1}} \times \mathbb{Z}_{p_2^{d_2}} \times \ldots \times \mathbb{Z}_{p_r^{d_r}}$ with $p_1 < p_2 < \cdots < p_r$. If*

$$\{p_i \mid (m,n) \,:\, i = 1 \ldots r\} \cup \left\{p_i \mid m \,:\ \mathbb{Z}_n \text{ acts on } \mathbb{Z}_{p_i^{d_i}} \text{ nontrivially}\right\} = \varnothing,$$

then $\chi_{\text{gen}}(G) = \infty$. Otherwise,

$$\chi_{\text{gen}}(G) = \min\{p_i \mid (m,n) \,:\, i = 1, \ldots, r\} \cup \left\{p_i \mid m \,:\ \mathbb{Z}_n \text{ acts on } \mathbb{Z}_{p_i^{d_i}} \text{ nontrivially}\right\}.$$

Proof. By induction on m. If m is prime, then Lemma 4.3 applies.

If $\mathbb{Z}_n$ acts trivialy on $\mathbb{Z}_{p_1^{d_1}}$ and $p_1 \nmid n$, then $\mathbb{Z}_{p_1^{d_1}} \leq Z(G)$. Thus,

$$\left(\mathbb{Z}_{p_1^{d_1}} \times \mathbb{Z}_{p_2^{d_2}} \times \cdots \times \mathbb{Z}_{p_r^{d_r}}\right) \rtimes \mathbb{Z}_n = \mathbb{Z}_{p_1^{d_1}} \times \left(\left(\mathbb{Z}_{p_2^{d_2}} \times \cdots \times \mathbb{Z}_{p_r^{d_r}}\right) \rtimes \mathbb{Z}_n\right).$$

By proposition 1.9, since $p_1^{d_1}$ and $mn/p_1^{d_1}$ are relatively prime,

$$\begin{aligned} &\chi_{\text{gen}}\left(\mathbb{Z}_{p_1^{d_1}} \times \left(\left(\mathbb{Z}_{p_2^{d_2}} \times \cdots \times \mathbb{Z}_{p_r^{d_r}}\right) \rtimes_\varphi \mathbb{Z}_n\right)\right) = \\ &= \min\left\{\chi_{\text{gen}}\left(\left(\mathbb{Z}_{p_2^{d_2}} \times \cdots \times \mathbb{Z}_{p_r^{d_r}}\right) \rtimes_\varphi \mathbb{Z}_n\right), \chi_{\text{gen}}\left(\mathbb{Z}_{p_1^{d_1}}\right)\right\}. \end{aligned}$$

By the induction hypothesis, we are done in this case.

If $\mathbb{Z}_n$ acts nontrivialy on $\mathbb{Z}_{p_1^{d_1}}$ or $p_1 \mid n$, we have that

$$\left(\mathbb{Z}_{p_1^{d_1}} \times \mathbb{Z}_{p_2^{d_2}} \times \cdots \times \mathbb{Z}_{p_r^{d_r}}\right) \rtimes_\varphi \mathbb{Z}_n / \left(\mathbb{Z}_{p_2^{d_2}} \times \cdots \times \mathbb{Z}_{p_r^{d_r}}\right) \cong \mathbb{Z}_{p_1^{d_1}} \rtimes_\varphi \mathbb{Z}_n.$$

Since $\mathbb{Z}_{p_1^{d_1}} \rtimes_\varphi \mathbb{Z}_n$ is non-cyclic,

$$\chi_{\text{gen}}\left(\left(\mathbb{Z}_{p_1^{d_1}} \times \mathbb{Z}_{p_2^{d_2}} \times \cdots \times \mathbb{Z}_{p_r^{d_r}}\right) \rtimes \mathbb{Z}_n\right) \leq \chi_{\text{gen}}\left(\mathbb{Z}_{p_1^{d_1}} \rtimes_\varphi \mathbb{Z}_n\right) = p_1.$$

It remains to show that $\chi_{\text{gen}}(\mathbb{Z}_m \rtimes \mathbb{Z}_n) \geq p_1$. Write $\mathbb{Z}_m \rtimes \mathbb{Z}_n = \langle x \rangle \rtimes \langle y \rangle$ let

$$H_1 \cup H_2 \cup \cdots \cup H_k = G$$

be irrudandent . If $x \in H_1 \cap H_2 \cap \cdots \cap H_k$ then

$$\langle y \rangle \cong G/\langle x \rangle = H_1/\langle x \rangle \cup \cdots \cup H_k/\langle x \rangle$$

Thus, for some i, $H_i/\langle x \rangle = G/\langle x \rangle$. Since $\langle x \rangle \leq H_i$, we have that $H_i = G$. Assume that $x \notin H_1$. Take $y \in H_1 \setminus (H_2 \cup \cdots \cup H_k)$ then at least two of the elements $y, x, xy, x^2y, x^3y, \ldots, x^{p-1}y$ are contained in the same H_i. If $x, x^jy \in H_i$ then $x, y \in H_i$ but since $y \in H_i$, we have that $i = 1$. But then $x \notin H_i$, a contradiction. If $x^jy, x^ty \in H_i$

with $j < t$ we have $x^{t-j} \in H_i$. Since $t - j < p_1$, $t - j$ and n are relatively prime. Thus, $x \in H_i$ and we obtain a contradiction as before. $\square$

Corollary 4.6. *$\chi_{\text{gen}}(D_{2n})$ is the minimal prime factor of n.*

Proof. $D_{2n} = \langle r, s : r^n = s^2 = e,\, srs = r^{-1}\rangle \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$. If n is even, then $\chi_{\text{gen}}(\mathbb{Z}_n \rtimes \mathbb{Z}_2) = 2$ by the preceding theorem. Otherwise, let p be the minimal prime factor of n, and let $\mathbb{Z}_{p^i}$ be the Sylow p-group of $\mathbb{Z}_n$. Let $y \in \mathbb{Z}_{p^i} \setminus \{e\}$ then $sys^{-1} = y^{-1} \neq y$. Thus, $\mathbb{Z}_2$ acts nontrivially on $\mathbb{Z}_{p^i}$ and the result follows from the preceding theorem. $\square$

Example 4.7. $\chi_{\text{gen}}(\mathbb{Z}_3 \rtimes \mathbb{Z}) = 3$, but $\mathbb{Z}_p \times \mathbb{Z}_p$ is not a quotient of $\mathbb{Z}_3 \rtimes \mathbb{Z}$ for any p.

Proof. As the automorphism group of $\mathbb{Z}_3$ has order 2, the action of $2\mathbb{Z}$ on $\mathbb{Z}_3$ by conjugation is trivial, that is, $2\mathbb{Z}$ is in the center of $\mathbb{Z}_3 \rtimes \mathbb{Z}$. Now, $\mathbb{Z}_3 \rtimes \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_3 \rtimes \mathbb{Z}_2 = D_6$. By Lemma 1.5 and Corollary 4.6, $\chi_{\text{gen}}(\mathbb{Z}_3 \rtimes \mathbb{Z}) \leq \chi_{\text{gen}}(D_6) = 3$. By Theorem 3.2, it remains to prove that $\mathbb{Z}_p \times \mathbb{Z}_p$ is not a quotient of $\mathbb{Z}_3 \rtimes \mathbb{Z}$ for any p.

Case 1: $p \neq 3$. Assume that $\mathbb{Z}_3 \rtimes \mathbb{Z}/H = \mathbb{Z}_p \times \mathbb{Z}_p$. Write $\mathbb{Z}_3 \rtimes \mathbb{Z} = \langle x\rangle \rtimes \langle y\rangle$. As $x^3 = e$, the order of xH divides both 3 and p, and is therefore 1. Thus, $x \in H$, that is, $\mathbb{Z}_3 \leq H$, and by the Third Isomorphism Theorem,

$$\mathbb{Z}/(H/\mathbb{Z}_3) = (\mathbb{Z}_3 \rtimes \mathbb{Z}/\mathbb{Z}_3)/(H/\mathbb{Z}_3) = \mathbb{Z}_3 \rtimes \mathbb{Z}/H = \mathbb{Z}_p \times \mathbb{Z}_p$$

is a quotient of a cyclic group, and thus cyclic, a contradiction.

Case 2: $p = 3$. Assume that $\mathbb{Z}_3 \rtimes \mathbb{Z}/H = \mathbb{Z}_3 \times \mathbb{Z}_3$. Write $\mathbb{Z}_3 \rtimes \mathbb{Z} = \langle x\rangle \rtimes \langle y\rangle$. Then $y^3 \in H$. Thus, $x^2y^3 = xy^3x^{-1} \in H$, and therefore $x^2 \in H$. It follows that $\mathbb{Z}_3 \leq H$, and we get a contradiction as in Case 1. $\square$

We have proved that the generating chromatic number of a nilpotent group is either ∞ or prime. One may wonder whether at least this can be generalized, for solvable groups at least. This is not the case.

Example 4.8. $\chi_{\text{gen}}(A_4) = 4$.

Proof. A_4 has four maximal subgroups of order 3 and one maximal subgroup of order 4. Together, these groups cover A_4. Thus, $\chi_{\text{gen}}(A_4) < 5$. On the other hand, no four of these groups cover A_4, since the identity elements belongs to them all, and thus the cardinality of their union is at most $1 + 3(3 - 1) + (4 - 1) = 10 < 12$. $\square$

It can be shown that $\chi_{\text{gen}}(S_4) = 3$ by verifying that $3 \leq \chi_{\text{gen}}(S_4)$ and using that $S_3 = D_6$ is a quotient of S_4.

5. Vector spaces and fields

Definition 5.1. The *generating chromatic number* of a vector space V, $\chi_{\text{gen}}(V)$, is the maximum number of colors k such that there is a monochromatic spanning set for each coloring of the elements of V in k colors. If no such maximal k exists, we set $\chi_{\text{gen}}(G) = \infty$.

The proofs of the following two lemmata are similar to those of Lemmata 1.5 and 1.2.

Lemma 5.2. *If a vector space U is a linear image of a vector space V, then $\chi_{\text{gen}}(V) \leq \chi_{\text{gen}}(U)$.* $\square$

Lemma 5.3. *Let V be a vector space . The following are equivalent:*

(1) $\chi_{\text{gen}}(V) \geq k$.
(2) *For each cover* $V = V_1 \cup \cdots \cup V_k$ *of* V *by subspaces, there is* i *with* $V_i = V$. □

The proof similar to Lemma 2.1 also establishes the following.

Lemma 5.4. *Let* $\mathbb{F}$ *be a finite field. The vector space* $\mathbb{F}^2$ *satisfies* $\chi_{\text{gen}}(\mathbb{F}^2) \leq |\mathbb{F}|$.

Proof. We find $|\mathbb{F}|+1$ proper subspaces that cover $\mathbb{F}^2$. For each $\alpha \in \mathbb{F}$, let $V_\alpha = \text{ span } \{(1,\alpha)\}$. Let $V = \text{ span } \{(0,1)\}$. To see that $\mathbb{F}^2 = V \cup \bigcup_{\alpha\in\mathbb{F}} V_\alpha$, let $(\alpha,\beta) \in \mathbb{F}^2$. If $\alpha = 0$ then $(\alpha,\beta) \in V$, and if not, then $(\alpha,\beta) \in \text{ span } \{(1,\alpha^{-1}\beta)\} \in V_{\alpha^{-1}\beta}$. □

Clearly, for vector spaces V of dimension 1, $\chi_{\text{gen}}(V) = \infty$.

The inequality $(\geq)$ in the following theorem was proved in Bialynicki-Birula–Browkin–Schinzel [1]. The other inequality was proved, e.g., in Khare [5].

Theorem 5.5 (Bialynicki-Birula–Browkin–Schinzel, Khare)**.** *Let* V *be a vector space of dimension* ≥ 2 *over a field* $\mathbb{F}$. *Then* $\chi_{\text{gen}}(V) = |\mathbb{F}|$ *if* $\mathbb{F}$ *is finite, and* ∞ *otherwise.*

Proof. $(\leq)$ By Lemmata 5.4 and 5.2.

$(\geq)$ Similar to the proof of Proposition 1.8. □

Theorem 5.6 (Bialynicki-Birula–Browkin–Schinzel [1])**.** *Let* $\mathbb{F}$ *be a field. For each coloring of the elements of* $\mathbb{F}$ *in finitely many colors, there a monochromatic set generating* $\mathbb{F}$ *as a field.*

Proof. Assume that there is a finite coloring c with no monochromatic generating set, with a finite, minimal number of colors. For each color i, let $\mathbb{F}_i$ be the subfield generated by the elements of color i. Let $\mathbb{H} = \bigcap_i \mathbb{F}_i$.

Since, as groups, $(\mathbb{F}_i,+) \leq (\mathbb{F},+)$ for all colors i, we have by Neumann's Theorem 1.10 that $(\mathbb{H},+)$ is of finite index in $(\mathbb{F},+)$.

If $\mathbb{H}$ is finite, then $\mathbb{F}$ is finite, and then the multiplicative group $\mathbb{F}^*$ is cyclic. In particular, $\mathbb{F}$ is genrated, as a field, by a single element, which in turn consitututes a monochromatic generating set, a contradiction.

If $\mathbb{H}$ is infinite, then as $\mathbb{F}$ and each $\mathbb{F}_i$ are vector spaces over the infinite field $\mathbb{H}$, we have by Thorem 5.5 that there is a monochromatic set generating $\mathbb{F}$ as a vector space of $\mathbb{H}$, and, in particular, as a field, a contradiction. □

6. Open problems

Tomkinson [8] provides a complete characterization of the generating chromatic number of arbitrary finite solvable groups.

An interesting direction may be to carry out a finer analysis of the case of *infinite* monochromatic numbers. To this end, define $\hat{\chi}_{\text{gen}}(G)$ as the minimal cardinal number of colors needed to color the elements of G such that there are no monochromatic generating sets. If this number is finite, then it is just $\chi_{\text{gen}}(G)+1$, but in the infinite case, this is the right definition. Some, but not all, of the proofs in this paper extend to the infinite case.

Credits. All results in this paper were initially proved by its authors, who were not aware of the earlier works cited in the present version of this paper. Initially, Theorem 5.5 was proved, chronologically earlier than the other results, by the second and third named authors. The other results were proved by the first and third named authors. Our Theorem 3.1 may be new.

References

[1] A. Bialynicki–Birula, J. Browkin, A. Schinzel, On the representation of fields as finite unions of subfields, Colloquium Mathematicum 7 (1959), 31–32.
[2] J. Cohn, On n-sum groups, Mathematica Scandinavica 75 (1994), 44–58.
[3] D. Dummit, R. Foote, Abstract Algebra, third edition, John Wiley and Sons, 2004.
[4] S. Haber and A. Rosenfeld, Groups as unions of proper subgroups, American Mathematical Monthly 66 (1959), 491–494.
[5] A. Khare, Vector spaces as unions of proper subspaces, Linear Algebra and its Applications 431 (2009), 1681–1686.
[6] B. Neumann, Groups covered by permutable subsets, Journal of the London Mathematical Society 29 (1954), 236–48.
[7] G. Scorza, I gruppi che possono pensarsi come somma di tre loro sottogruppi, Bollettino Unione Matematica Italiana 5 (1926), 216–218.
[8] M. Tomkinson, Groups as the union of proper subgroups, Mathematica Scandinavica 81 (1997), 189–198.

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